

On the fulfillment of the energy conservation law in mathematical models of evolution of single spherical bubble

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Abstract

The problem of the evolution of a single spherical bubble in an infinite liquid is considered, as the result of a variation of the pressure in the liquid at an infinite distance from the bubble. It has been assumed the bubble is filled with vapor from the surrounding liquid or insoluble gas. The question of the fulfillment of the integral energy conservation law is investigated using different ways of describing the hydrodynamic and heat and mass exchange processes in both the bubble and surrounding liquid and at the bubble interface. Kinetic and internal energy of vapor (gas) in the bubble, kinetic and internal energy of the liquid, and energy of surface tension are taken into account in the energy balance. The liquid is assumed to be incompressible, viscous and heat-conducting, the vapor (gas) to be nonviscous, heat-conducting and obeying the Clapeyron equation. Thermal–physical properties, exclusive of specific heats, are allowed to be temperature-dependent. For the above suppositions and assumptions, a mathematical model ensuring exact fulfillment of the integral energy conservation law has been developed. It has been shown that the conservation integral can be fulfilled by the given model. As simplified variants of the principal model, models of the uniform bubble and pressure uniform bubble, have been proposed which ensure the exact fulfillment of the integral energy balance disregarding the relatively small vapor kinetic energy. A relation defining the imbalance in the integral energy conservation law for some often-used extra simplifications has been derived.

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1. Introduction

Cavitation involving a single spherical bubble with variation of pressure in the surrounding liquid is a model problem of fundamental importance for the understanding of processes associated with cavitation. By now a number of approaches to solving the problem using mathematical models have been proposed [1–4]. To one extent or another all these approaches are based on various simplifications of real physical processes. It should be noted that in the problem at hand correctly taking into account some physical effects, e.g., liquid viscosity or surface tension, would lead

to the necessity of simultaneous introduction of modifications in several equations. This is especially true where phase transitions that influence the kinematic, dynamic and thermal aspects of the problem are taken into account. Clearly, the simplified description of such processes under these conditions is not uniquely defined and can be made in various ways. It should also be noted that the correctness of a given simplification depends, among other things, on the aim of an investigation. For example, in studies of bubble dynamics, models assuming either the constancy of temperatures in the bubble and surrounding liquid or the complete absence of heat exchange between the bubble and liquid, are of widespread occurrence [2,3]. It is obvious that for any application sensitive to varying temperature in the bubble, both suppositions mentioned are not acceptable.

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Nomenclature

A	work done by external pressure forces
c	liquid specific heat capacity
c_V	heat capacity of the vapor at constant volume
c_p	specific heat capacity at constant pressure
K	kinetic energy of the liquid
E	internal energy
E_σ	surface energy
i	specific enthalpy
j	mass of vapor condensing per unit time on unit area of the interface
F	parameter in Eq. (38)
q	heat flux
p	pressure
r	radial coordinate
R	radius of the bubble
R_V	gas constant
S	area of the bubble surface
t	current time
T	temperature
u	velocity
U	liquid velocity at the bubble boundary

V	volume of the bubble
w	vapor velocity at the bubble

Greek symbols

α	parameter in Eqs. (53)–(56)
β	parameter in Eq. (38)
γ	ratio of specific heats
ε	specific internal energies
λ	thermal conductivity coefficient
μ	dynamic coefficient of viscosity
θ	temperature increment
ρ	density
σ	coefficient of surface tension
τ	friction stress
Φ	dissipative function
ψ	heat of phase transition

Subscripts

s	steam–water interface
v	vapor
0	initial value

Therefore, the comparison of the quality of mathematical models based on different ways of simplification of processes in the problem at hand is not uniquely defined. In this work a new approach on the basis of the energy conservation integral law is proposed. It should be noted that the currently available models including those that were studied in detail [1] do not ensure the exact fulfillment of the energy integral balance. Hence the first objective of this article is to obtain such a mathematical model on the basis of general laws of hydromechanics and thermodynamics with heat-and-mass exchange, which could ensure the exact fulfillment of the integral law for the problem at hand with the minimum number of assumptions. The second objective is to assess the effects of an imbalance in integral energy conservation that can arise for certain typical deviations from the basic mathematical model.

2. The formulation of the problem and some intermediate relationships

The process of compression or expansion of a vapor or gas bubble (in the absence of dissolution) is considered at a change of pressure in the surrounding liquid. For definiteness, we will examine the case of compression of the vapor bubble. At the initial time moment, the liquid and vapor are in equilibrium and have temperature T_0 . We take the following physical hypotheses and assumptions.

- (1) As in the majority of works, the problem is considered under the spherical-symmetry assumption. Prob-

able reasons of a deviation of the bubble shape from spherical have been investigated in [5].

- (2) The liquid is assumed to be incompressible, viscous (Newtonian liquid), and heat-conducting. Models including compressibility of the liquid have been described in [6].
- (3) The vapor in the bubble is considered as nonviscous, heat-conducting and describable by the Clapeyron equation.
- (4) At the interface, the vapor temperature is set equal to the liquid temperature and the vapor pressure is made to correspond to the liquid temperature on the saturation curve.
- (5) In contrast to other models, the viscosity and surface tension are assumed to be variable (for example, depending on temperature). As a result, additive terms will appear in some of the relevant equations. The influence of viscosity and surface tension on the dynamics of bubbles at constant values of pertinent coefficients was investigated in [7]. Thermal conductivity of liquid and vapor are also assumed to be variable, but heat capacity of liquid and vapor are considered to be constant.

In the case of spherical symmetry, the equation of motion and the solution of a continuity equation for Newtonian incompressible liquid can be written in the form [8]:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) = - \frac{\partial p}{\partial r} + \frac{\partial \tau}{\partial r} + \frac{2}{r} \left(\tau - \frac{2\mu u}{r} \right), \quad \tau = 2\mu \frac{\partial u}{\partial r} \quad (1)$$

$$u = \frac{1}{r^2} UR^2 \tag{2}$$

Here t, r are the time and radial coordinates; u, ρ are liquid velocity and density; p is the ambient pressure; μ is the dynamic coefficient of viscosity; τ is the friction stress; R is the radius of the bubble; U is the liquid velocity at the bubble boundary, related to the velocity of motion of the bubble boundary as follows:

$$\frac{dR}{dt} = U - \frac{j}{\rho} \tag{3}$$

Here j is the mass of vapor condensing per unit time on unit area of the interface. Integrating Eq. (1) with respect to radius from $r = R$ to $r = \infty$ and considering that $\tau \rightarrow 0$ with $r \rightarrow \infty$, we obtain, using (2) and (3), the following equation:

$$\rho \left(R \frac{dU}{dt} + \frac{3}{2} U \frac{dR}{dt} \right) = p_R - p_\infty - \tau_R - \Omega + \frac{1}{2} jU, \tag{4}$$

$$\Omega = 12UR^2 \int_R^\infty \mu r^{-4} dr$$

Here p_∞ is the liquid pressure at an infinite distance from the bubble, p_R and τ_R are liquid pressure and friction stress at the interface.

We use the energy equation for the liquid surrounding the bubble:

$$\rho c \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \lambda \frac{\partial T}{\partial r} \right) + \Phi, \tag{5}$$

$$\Phi = 12\mu U^2 R^4 r^{-6}$$

Here c, λ, T are the liquid specific heat capacity, thermal conductivity coefficient and temperature. Φ is a dissipative function. Considering Eq. (2), we rearrange Eq. (5) to the form:

$$\rho c \frac{\partial}{\partial t} (r^2 \theta) + \rho c UR^2 \frac{\partial \theta}{\partial r} = - \frac{\partial}{\partial r} (r^2 q) + r^2 \Phi, \tag{6}$$

$$q = -\lambda \frac{\partial \theta}{\partial r}$$

Here $\theta = T - T_0$, and q is the specific heat flux due to heat conduction in the liquid.

Consider now processes within the bubble. The Clapeyron equation and the motion, continuity and energy equations for an inviscid perfect gas have the form:

$$p_V = \rho_V R_V T_V \tag{7}$$

$$\rho_V \frac{\partial w}{\partial t} + \rho_V w \frac{\partial w}{\partial r} = - \frac{\partial p_V}{\partial r} \tag{8}$$

$$\frac{\partial \rho_V}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho_V w r^2) = 0 \tag{9}$$

$$\rho_V c_p \left(\frac{\partial T_V}{\partial t} + w \frac{\partial T_V}{\partial r} \right) = \frac{\partial p_V}{\partial t} + w \frac{\partial p_V}{\partial r} - \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 q_V), \tag{10}$$

$$q_V = -\lambda_V \frac{\partial T_V}{\partial r}$$

Here $p_V, \rho_V, T_V, R_V, \lambda_V, c_p, w, q_V$ denote pressure, density, temperature, gas constant, thermal conductivity coefficient, specific heat capacity at constant pressure, velocity and heat flux. We note for later reference that from Eqs. (7), (9), (10) the following formula can be derived:

$$\frac{1}{\gamma - 1} \int_V \frac{\partial p_V}{\partial t} dV + \frac{\gamma}{\gamma - 1} p_{VS} \frac{dV}{dt} + c_p J T_{VS} = \int_V w \frac{\partial p_V}{\partial r} dV - S q_{VS} \tag{11}$$

Here V, S are the volume and area of the bubble surface; γ is the specific heat ratio; T_{VS} and p_{VS} are the vapor temperature and pressure at the bubble boundary; $J = jS$, q_{VS} is the specific heat flux from the vapor to the bubble surface.

Now consider the conditions to be applied at the interface. Using relationships for the interface from reference [2] we can write:

$$p_R = p_{VS} + \tau_R - \frac{2\sigma}{R} - j(U - w_S) \tag{12}$$

$$q_S - q_{VS} = j\psi + \frac{j}{\rho} \frac{2\sigma}{R} - \frac{d\sigma}{dt} + \frac{1}{2} j(U - w_S)^2 \tag{13}$$

$$\psi = i_{VS} - i_S^* \tag{14}$$

Here σ is the coefficient of surface tension, q_S is the specific heat flux through the interface to liquid, w_S, i_{VS} are the vapor velocity and enthalpy at the interface, i_S^* is the liquid enthalpy at the temperature and vapor pressure at the interface (note that this is not equal to the liquid enthalpy at the interface). From Eqs. (13), (14) it follows that ψ is a heat of phase transition at a flat surface for infinitely small velocities of the liquid and vapor.

Now consider some thermodynamic aspects of the problem. At constant specific heat capacities, the specific internal energies and specific enthalpies of the liquid and vapor are defined by the formulas:

$$\varepsilon = c\theta + \varepsilon_0, \quad i = \varepsilon + \frac{p}{\rho}, \quad \varepsilon_V = c_V T_V + \varepsilon_{V0}, \tag{15}$$

$$i_V = \varepsilon_V + \frac{p_V}{\rho_V} = c_p T_V + \varepsilon_{V0}$$

Here c_V is the heat capacity of the vapor at constant volume. The constants ε_0 and ε_{V0} are correlated with each other through a heat of phase transition (note that for the problem at hand one of these constants may be assigned arbitrarily). Setting ψ_0 to be the heat of phase transition at the initial temperature and pressure values, and using Eqs. (14) and (15), one can obtain:

$$\varepsilon_{V0} = \varepsilon_0 + \psi_0 - c_p T_0 + \frac{p_{V0}}{\rho} \tag{16}$$

Here p_{V0} is the initial vapor pressure. Using the designation $\theta_S = T_S - T_0$, we rearrange Eq. (14) into the form:

$$\psi = \psi_0 + \theta_S (c_p - c) - \frac{(p_{VS} - p_{V0})}{\rho} \tag{17}$$

3. Integral energy balance

We will introduce designations:

$$K = \int_R^\infty \frac{1}{2} \rho u^2 4\pi r^2 dr = 2\pi\rho U^2 R^3, \quad E = \int_R^\infty \rho \varepsilon 4\pi r^2 dr,$$

$$K_V = \int_V \frac{1}{2} \rho_V w^2 dV,$$

$$E_V = \int_V \rho_V \varepsilon_V dV, \quad E_\sigma = \sigma S, \quad A = -USp_\infty$$
(18)

Here K , E are the kinetic and internal energies of the liquid; K_V , E_V are the kinetic and internal energies of the vapor; E_σ is the surface energy; A is the work done by external pressure forces. Using Eqs. (3), (4), (12), we obtain:

$$\frac{dK}{dt} = 2\pi\rho \frac{d}{dt}(U^2 R^3) = \rho US \left(R \frac{dU}{dt} + \frac{3}{2} U \frac{dR}{dt} \right)$$

$$= USp_{VS} - US\Omega - US \frac{2\sigma}{R} - \frac{1}{2} JU^2 + JUw_S + A$$
(19)

For the vapor internal energy, using Eqs. (7), (9), (11), (13)–(15), we can obtain upon rearrangement:

$$\frac{dE_V}{dt} = \frac{1}{\gamma - 1} \int_V \frac{\partial p_V}{\partial t} dV + \rho_{VS}(\varepsilon_{VS} - \varepsilon_{V0}) \frac{dV}{dt} - \varepsilon_{V0} J$$

$$= -p_{VS} \frac{dV}{dt} - Sq_{VS} + \int_V w \frac{\partial p_V}{\partial r} dV - i_{VS} J$$

$$= -p_{VS} \frac{dV}{dt} - Sq_S + \int_V w \frac{\partial p_V}{\partial r} dV - i_{VS} J + J\psi$$

$$+ S \left(\frac{j}{\rho} \frac{2\sigma}{R} - \frac{d\sigma}{dt} \right) - JUw_S + \frac{1}{2} JU^2 + \frac{1}{2} Jw_S^2$$

$$= -SU p_{VS} - Sq_S + \int_V w \frac{\partial p_V}{\partial r} dV$$

$$+ S \left(\frac{j}{\rho} \frac{2\sigma}{R} - \frac{d\sigma}{dt} \right) - JUw_S + \frac{1}{2} JU^2 + \frac{1}{2} Jw_S^2 - J\varepsilon_S$$
(20)

Here ε_S is the specific internal energy of the liquid at the interface.

For the liquid internal energy, with the use of Eqs. (6), (15), under the assumption that the heat flux for the radius approaching infinity tends to zero faster than the radius squared, we shall obtain:

$$\frac{dE}{dt} = 4\pi \left(\int_R^\infty \rho \frac{\partial \varepsilon}{\partial t} r^2 dr - \rho R^2 \varepsilon_S \frac{dR}{dt} \right)$$

$$= 4\pi \left(-\rho UR^2 \varepsilon_0 + R^2 q_S + \frac{1}{4\pi} US\Omega - \rho R^2 \varepsilon_S \left(\frac{dR}{dt} - U \right) \right)$$

$$= Sq_S + J\varepsilon_S + US\Omega - \rho US\varepsilon_0$$
(21)

Combining Eqs. (19)–(21), we shall obtain the following expression:

$$\frac{dK}{dt} + \frac{dE}{dt} + \frac{dE_V}{dt} + \frac{dE_\sigma}{dt} - A + \rho US\varepsilon_0$$

$$= \int_V w \frac{\partial p_V}{\partial r} dV + \frac{1}{2} Jw_S^2$$
(22)

For kinetic energy of the vapor in the bubble:

$$\frac{dK_V}{dt} = 2\pi \int_0^R \frac{\partial(\rho_V w^2)}{\partial t} r^2 dr + \frac{1}{2} Sw_S^2 \rho_{VS} \frac{dR}{dt}$$

$$= 2\pi \int_0^R \frac{\partial(\rho_V w^2)}{\partial t} r^2 dr + \frac{1}{2} S\rho_{VS} w_S^3 - \frac{1}{2} Jw_S^2$$
(23)

From Eqs. (8), (9), upon rearrangement, one can derive the following equation:

$$r^2 \frac{\partial}{\partial t} (\rho_V w^2) + \frac{\partial}{\partial r} (\rho_V w^3 r^2) = -2wr^2 \frac{\partial p_V}{\partial r}$$
(24)

Substituting Eq. (24) in (23), we obtain:

$$\frac{dK_V}{dt} = - \int_0^R 4\pi r^2 w \frac{\partial p_V}{\partial r} dr - \frac{1}{2} Jw_S^2 = - \int_V w \frac{\partial p_V}{\partial r} dV - \frac{1}{2} Jw_S^2$$
(25)

Combining now Eqs. (22) and (25), we obtain the integral energy equation in the following final form:

$$\frac{dK}{dt} + \frac{dK_V}{dt} + \frac{dE}{dt} + \frac{dE_V}{dt} + \frac{dE_\sigma}{dt} = A - \rho US\varepsilon_0$$
(26)

It should be noted that in the spherical symmetry conditions, the product US is equal to the volumetric flow rate through any closed surface including the centre of symmetry, including also a spherical surface with an infinite radius. And so, Eq. (26) reveals that an increase in the total energy of system equals the work done by impressed forces plus the internal energy arriving in the system along with mass of the liquid inflowing through the infinitely distant boundary of the system. It is obvious that Eq. (26) is an exact expression of the integral law of energy conservation for the problem at hand in the context of the physical man-made assumptions.

4. The fundamental mathematical model

When deriving Eq. (26), the equations were used which, upon some formal rearrangement, may be presented as follows:

$$\rho \left(R \frac{dU}{dt} + \frac{3}{2} U^2 \right) = p_{VS} - p_\infty - \frac{2\sigma}{R} - 12UR^2 \int_R^\infty \mu r^{-4} dr$$

$$+ jU + jw_S$$
(27)

$$\rho c \frac{\partial}{\partial t} (r^2 \theta) + \rho c UR^2 \frac{\partial \theta}{\partial r} = \frac{\partial}{\partial r} \left(r^2 \lambda \frac{\partial \theta}{\partial r} \right) + 12\mu U^2 R^4 r^{-4}$$
(28)

$$p_V = \rho_V R_V T_V$$
(29)

$$\rho_V \frac{\partial w}{\partial t} + \rho_V w \frac{\partial w}{\partial r} = - \frac{\partial p_V}{\partial r}$$
(30)

$$\frac{\partial \rho_V}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho_V w r^2) = 0$$
(31)

$$\rho_V c_p \left(\frac{\partial T_V}{\partial t} + w \frac{\partial T_V}{\partial r} \right) = \frac{\partial p_V}{\partial t} + w \frac{\partial p_V}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \lambda_V \frac{\partial T_V}{\partial r} \right) \quad (32)$$

$$\frac{dR}{dt} = U - \frac{j}{\rho} \quad (33)$$

$$q_s - q_{vs} = j\psi + \frac{j}{\rho} \frac{2\sigma}{R} - \frac{d\sigma}{dt} - jUw_s + \frac{1}{2}jU^2 + \frac{1}{2}jw_s^2 \quad (34)$$

$$\psi = \psi_0 + \theta_s(c_p - c) - \frac{(p_{vs} - p_{v0})}{\rho} \quad (35)$$

$$\frac{d}{dt} \int_V \rho_V dV = -jS, \quad (36)$$

In the above system, Eqs. (28)–(32), (36) are general for hydrodynamics, therefore, all the specific character of the problem at hand and hence of the mathematical model belongs to Eqs. (27), (33)–(35). Here, Eq. (27) derived from Eqs. (1)–(3), (12) may be considered as generalization of the well known Rayleigh–Plesset equation [3] for the case of available phase transitions and the variable viscosity coefficient of the liquid.

Eqs. (27)–(36) describe evolution of both vapor and gas bubbles. Since the relation between the pressure and temperature of the vapor at the bubble surface through the saturation curve has not been used, for a change in these equations from the case of the vapor to the case of insoluble gas, it would be sufficient to put $j = 0$. As this takes place the constants ε_0 and ε_{v0} become independent variables.

It is easy to check that the system of Eqs. (27)–(36) is not sufficient to define the evolution of the vapor bubble, as its number of unknown variables is more by one than the number of equations. To complete the mathematical model in this case, the system of Eqs. (27)–(36) must be supplemented with an equation for the saturation curve:

$$p_{vs} = f(T_s) \quad (37)$$

Eq. (37) is assignable in different forms, e.g., in the form of the Clapeyron–Clausius equation [9]. It may also take into account the effect of the surface curvature on the saturation vapor pressure [10]. It is however clear that the fulfillment of the integral energy balance will not be susceptible to the specific form of Eq. (37).

We note for the following discussion that the complete system of Eqs. (27)–(37) includes algebraic equations, ordinary differential equations, four partial differential equations and two integro-differential equations. Along with boundary and initial conditions, that are sufficiently evident for the problem at hand, this system of equations constitutes the mathematical model that will be adopted as the fundamental model. The model is complicated, and only numerical solutions can be obtained for it. However, in a special case, when the pressure p_∞ is constant, the basic model has a specific analytical solution in the form of a conservation integral, i.e., there is a function F which is a certain combination of relevant parameters and independent of time. Yield the function, it would be simplest to integrate Eq. (26) over time and take into account that

the integral of the product US , taken with inverse sign, is equal to a volume of the liquid having come to the system through the outer infinitely distant boundary, and it may be expressed through a variation in the bubble volume and mass. Cutting out intermediate manipulations, we give the final result:

$$\begin{aligned} F &= 2\pi\rho U^2 R^3 + 2\pi \int_0^R \rho_V w^2 r^2 dr + \sigma S + 4\pi\rho c \int_R^\infty \theta r^2 dr \\ &\quad + V \left(\frac{\bar{p}_V}{\gamma - 1} + p_\infty \right) + \beta m_V = \text{const} \\ \bar{p}_V &= \frac{1}{V} \int_V p_V dV, \quad m_V = \int_V \rho_V dV, \\ \beta &= \psi_0 - c_p T_0 + \frac{p_{v0} - p_\infty}{\rho} \end{aligned} \quad (38)$$

5. The model of the pressure uniform bubble

One radical way to simplify the problem is the introduction of a supposition of pressure uniformity throughout the volume of the bubble (homobaricity) [1]. Spatial nonuniformities of pressure in gas are small as compared to the average pressure when the velocity is much less than the sound speed [11]. With the use of the homobaricity supposition, the momentum Eq. (30) is no longer present in the system of equations, and the number of partial differential equations is diminished by unity. And using the approach [12] the number of partial differential equations may be further diminished by unity as follows. First, using Eq. (31) the velocity is eliminated from Eq. (32), then using Eq. (29) the temperature is excluded from the equation obtained. As a result, instead of Eqs. (31), (32), we obtain the equation for the vapor density:

$$r^2 \frac{\partial \rho_V}{\partial t} = \frac{1}{c_p} \frac{\partial}{\partial r} \left(\frac{r^2 \lambda_V}{\rho_V} \frac{\partial \rho_V}{\partial r} \right) + \frac{1}{3\gamma p_V} \frac{dp_V}{dt} \frac{\partial}{\partial r} (\rho_V r^3) \quad (39)$$

Integrating Eq. (39) with respect to radius from 0 to the bubble radius R , we shall obtain upon rearrangement:

$$\frac{dp_V}{dt} = -\frac{3(\gamma - 1)}{R} \left(\frac{\lambda_V p_V}{\rho_{Vs}^2 R_V} \frac{\partial \rho_V}{\partial r} \Big|_{r=R} + j c_p T_{vs} + \frac{\gamma p_V}{\gamma - 1} \frac{dR}{dt} \right) \quad (40)$$

It is easy to demonstrate that Eqs. (39), (40) offer also the automatic fulfillment of integral mass balance Eq. (36).

Although the rearrangement proposed above makes it possible to exclude the vapor velocity from the problem, it can be easily calculated for any time from the continuity equation. The velocity distribution may be formally used for calculation of the vapor kinetic energy and substitution into Eq. (26) however the equality (26) will not be fulfilled in such an approach. The point is that to obtain Eq. (26), it is necessary that the velocity and pressure of the vapor would also satisfy the equation of motion (30), but this is impossible for the model of the homobaric bubble. Thus,

the supposition as to pressure uniformity in the bubble is incompatible with the exact fulfillment of the integral energy Eq. (26).

For the problem in hand, the kinetic energy of the vapor or gas in the bubble makes up, as a rule, a small portion of the total energy of the system. If this energy can be neglected, the energy conservation law will obviously have the following form:

$$\frac{dK}{dt} + \frac{dE}{dt} + \frac{dE_V}{dt} + \frac{dE_\sigma}{dt} = A - \rho US \varepsilon_0 \quad (41)$$

As it follows from Eq. (22), the gas bubble ($j=0$) main equations, under the assumption of pressure uniformity, result in the identical fulfillment of Eq. (41) without any modifications. In the case of the vapor bubble ($j \neq 0$), it can be that Eq. (41) would be fulfilled, if it is presupposed in addition that at the interface the vapor velocity is much less than the liquid velocity and, as a result, the value of w_S in Eqs. (12), (13) may be neglected. Note, for the gas bubble $w_S = U$, however, with an available phase transition, a situation where $|w_S| \ll |U|$, is quite possible.

Let us take the assumption as to the pressure uniformity through the bubble volume, and the conditions $|w_S| \ll |U|$. Then the system of equations for the pressure uniform bubble model will have the form:

$$\rho \left(R \frac{dU}{dt} + \frac{3}{2} U^2 \right) = p_V - p_\infty - \frac{2\sigma}{R} - 12UR^2 \int_R^\infty \mu r^{-4} dr + jU \quad (42)$$

$$\rho c \frac{\partial}{\partial t} (r^2 \theta) + \rho c UR^2 \frac{\partial \theta}{\partial r} = \frac{\partial}{\partial r} \left(r^2 \lambda \frac{\partial \theta}{\partial r} \right) + 12\mu U^2 R^4 r^{-4} \quad (43)$$

$$p_V = \rho_V R_V T_V \quad (44)$$

$$r^2 \frac{\partial \rho_V}{\partial t} = \frac{1}{c_p} \frac{\partial}{\partial r} \left(r^2 \lambda_V \frac{\partial \rho_V}{\partial r} \right) + \frac{1}{3\gamma p_V} \frac{dp_V}{dt} \frac{\partial}{\partial r} (\rho_V r^3) \quad (45)$$

$$\frac{dp_V}{dt} = -\frac{3(\gamma-1)}{R} \left(\frac{\lambda_V p_V}{\rho_V^2 R_V} \frac{\partial \rho_V}{\partial r} \Big|_{r=R} + j c_p T_{VS} + \frac{\gamma p_V}{\gamma-1} \frac{dR}{dt} \right) \quad (46)$$

$$\frac{dR}{dt} = U - \frac{j}{\rho} \quad (47)$$

$$q_S - q_{VS} = j\psi + \frac{j}{\rho} \frac{2\sigma}{R} - \frac{d\sigma}{dt} + \frac{1}{2} j U^2 \quad (48)$$

$$\psi = \psi_0 + \theta_S (c_p - c) - \frac{(p_V - p_{V0})}{\rho} \quad (49)$$

$$p_V = f(T_S) \quad (50)$$

It is easy to confirm that a consequence of Eqs. (42)–(50) is the integral energy conservation law without regard to the vapor kinetic energy in the form of Eq. (41).

6. The model of the uniform bubble

The next natural step towards simplifying the mathematical model is the introduction of the assumption that not only pressure but temperature (and therefore, density)

is also uniform throughout the bubble volume. Using this assumption for the further simplification of the homobaric model, on some rearrangement we obtain the model of the uniform bubble, consisting of Eqs. (42)–(44), (47), (49), (50) and equations:

$$\frac{1}{\gamma-1} \frac{d(p_V V)}{dt} + p_V \frac{dV}{dt} = -S q_S + J\psi + \frac{J}{\rho} \frac{2\sigma}{R} - S \frac{d\sigma}{dt} + \frac{1}{2} j U^2 - c_p J T_V \quad (51)$$

$$\frac{d}{dt} (\rho_V V) = -J \quad (52)$$

The model of the uniform bubble contains only one partial differential equation – the energy equation for the liquid (43). It is interesting to note that the uniform bubble model is equivalent to the model of the pressure uniform bubble from the viewpoint of the integral energy balance. It is easy to check that the uniform bubble model cannot implement the Eq. (26) for the total energy balance, but provides energy balance without regard to the vapor kinetic energy (41).

7. On some of a violation of the integral energy balance

The pressure uniformity assumption or bubble uniformity assumption are widely used for various models. However, the energy balance in Eq. (41) may then not be fulfilled because of an incomplete consideration of phase transitions in the integral equation of motion of the liquid and in equations for pressure drop and heat flux at the interface. We will write these equations in the form:

$$\rho \left(R \frac{dU}{dt} + \frac{3}{2} U^2 \right) = p_R - p_\infty - \tau_R - \Omega + \alpha_1 2jU \quad (53)$$

$$p_R = p_V + \tau_R - \frac{2\sigma}{R} - \alpha_2 jU \quad (54)$$

$$q_S - q_{VS} = j\psi + \alpha_3 \frac{j}{\rho} \frac{2\sigma}{R} - \frac{d\sigma}{dt} + \alpha_4 \frac{1}{2} j U^2 \quad (55)$$

At $\alpha_i = 1$ Eqs. (53)–(55) are exact equations within the given limits of this physical model. However, in specific conditions, some terms in the equations are small and may often be neglected with available simplified approaches. Such simplified equations can be derived from Eqs. (53)–(55) by putting specified coefficients α_i equal to zero. If Eqs. (53)–(55) are used for derivation of the equation for the integral energy balance, instead of Eq. (41) we can obtain:

$$\begin{aligned} \frac{dK}{dt} + \frac{dE}{dt} + \frac{dE_V}{dt} + \frac{dE_\sigma}{dt} - A + \rho US \varepsilon_0 \\ = J \left(\frac{2\sigma}{\rho R} (\alpha_3 - 1) + U^2 \left(2\alpha_1 - \alpha_2 + \frac{1}{2} \alpha_4 - \frac{3}{2} \right) \right) \end{aligned} \quad (56)$$

The right-hand side of Eq. (56) reveals what error would arise in the integral energy balance for various simplifications of Eqs. (53)–(55).

We also note that the above mathematical models ensuring the exact fulfillment of the integral energy balance in

the form of Eq. (26) or (41) require a numerical realization. In doing so, the energy balance can break down to one degree or another because of errors in difference approximations. Nevertheless, a check of the energy balance will be very useful in this case too, because, the value of imbalance can serve as an efficient generalized indicator of computational errors.

8. Conclusions

Based on concretization of general laws of hydrodynamics, thermodynamics and heat-and-mass exchange, a mathematical model has been obtained that ensures the exact fulfillment of the system's integral energy conservation law for the problem at hand. In doing so, the system energy includes kinetic energy of the liquid and vapor, internal energy of the liquid and vapor and surface energy. All thermal and physical properties of the liquid and vapor, excepting specific heat capacities are taken to be temperature dependent. It has been revealed that at a constant pressure in the liquid at an infinite distance from the bubble, the basic mathematical model has an analytic solution in the form of the conservation integral.

It has been revealed that the supposition as to the bubble homobaricity is inconsistent with the exact fulfillment of total integral energy balance. The models of homobaric and uniform bubbles have been proposed providing the exact fulfillment of the integral energy balance without regard to vapor kinetic energy. A rearrangement allowing a reduction in the number of partial differential equations

for the model of the pressure uniform bubble has also been proposed. A relation defining an imbalance in integral energy for some often-used simplifications of basic equations has been derived.

The results obtained apply to both the vapor and gas bubble cases in the absence of dissolution.

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